A physicist's proof of the Lagrange–Good multivariable inversion formula

2003 J. Phys. A: Math. Gen. 36 9471

(http://iopscience.iop.org/0305-4470/36/36/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **36** (2003) 9471–9477 PII: S0305-4470(03)57260-2

A physicist's proof of the Lagrange–Good multivariable inversion formula

Abdelmalek Abdesselam

LAGA, Institut Galilée, CNRS UMR 7539, Université Paris XIII, Avenue J B Clément, F93430 Villetaneuse, France

E-mail: abdessel@math.univ-paris13.fr

Received 5 December 2002, in final form 4 July 2003
Published 27 August 2003
Online at stacks.iop.org/JPhysA/36/9471

Abstract
We provide a new proof, inspired by quantum field theory, of the Lagrange–Good multivariable inversion formula.

PACS numbers: 02.10.Ox, 03.70.+k

1. Introduction and sketch of the proof

The Lagrange inversion formula [13] is one of the most useful tools in enumerative combinatorics (see [9, 17]). Various efforts have been devoted to finding purely combinatorial proofs and generalizations of this formula. One of the many such generalizations is the extension from the one variable to the multivariable case. Early contributions in this direction can be found in [4, 11, 14, 16, 18], but the credit for the discovery of the general multivariable formula is usually attributed to the mathematical statistician I J Good [8]. We recommend [7] for a clear and thorough presentation as well as for more complete references. Quoted from [7] the Lagrange–Good formula says the following.

Theorem 1. *Let the formal power series $f_1, \ldots, f_m$ in the variables $x_1, \ldots, x_m$ be defined by*

$$f_i = x_i g_i(f_1, \ldots, f_m) \qquad 1 \leqslant i \leqslant m \tag{1}$$

for some formal power series $g_i(y_1, \ldots, y_m)$. Then the coefficient of

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \qquad \text{in} \quad \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_m^{k_m}}{k_m!} g_1^{n_1}(x_1, \ldots, x_m) \ldots g_m^{n_m}(x_1, \ldots, x_m)$$

is equal to the coefficient of

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \qquad \text{in} \quad \frac{f_1^{k_1}}{k_1!} \cdots \frac{f_m^{k_m}}{k_m!} \times \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f_1, \ldots, f_m))}$$

where

$$g_{ij}(y_1,\ldots,y_m) \stackrel{\text{def}}{=} \frac{\partial g_i}{\partial y_j}(y_1,\ldots,y_m). \tag{2}$$

The odd-looking determinant in the denominator was probably one of the reasons why this general formula was not discovered until [8]. However, a similar determinantal denominator appeared earlier in the classical MacMahon master theorem [15]. This is no coincidence since the latter is well known to be the linear special case of the Lagrange–Good formula. Our proof will rely on the toy model of quantum field theory introduced in [1] and which is related to an earlier formula of Gallavotti [6] for the Lindstedt series in KAM theory, in order to express the compositional inverse of a power series in the multivariable setting. Our derivation of the Lagrange–Good formula will follow from this representation of the formal inverse by straightforward and quite natural field of theoretical computations which will, in particular, explain the determinantal denominator as a 'normalization factor for a probability measure'.

Now let us quickly sketch our proof from a *functional* (as opposed to diagrammatic) point of view. Let $\overline{\phi}_1,\ldots,\overline{\phi}_m,\phi_1,\ldots,\phi_m$ denote the components of a complex Bosonic field to be integrated over $\mathbb{C}^m$ with the measure

$$\mathrm{d}\overline{\phi}\,\mathrm{d}\phi \stackrel{\text{def}}{=} \prod_{i=1}^{m}\left(\frac{\mathrm{d}(\mathrm{Re}\,\phi_i)\,\mathrm{d}(\mathrm{Im}\,\phi_i)}{\pi}\right). \tag{3}$$

We introduce an unnormalized correlation function

$$\int_{\mathbb{C}^m} \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\Omega(\phi)\,\mathrm{e}^{-\overline{\phi}\phi+\overline{\phi}xg(\phi)} \tag{4}$$

where $\overline{\phi}xg(\phi) \stackrel{\text{def}}{=} \sum_{i=1}^{m}\overline{\phi}_i x_i g_i(\phi)$, and Ω is a function of the holomorphic component ϕ only. Consider the transformation $V(\phi) \stackrel{\text{def}}{=} \phi - xg(\phi)$ whose ith component is given by $\phi_i - x_i g_i(\phi)$, and perform the change of variables $\overline{\phi}\to\overline{\phi}$, $\phi\to V(\phi)$ in the integral (4). The result is

$$\int_{\mathbb{C}^m} \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\, J(V^{-1})(\phi)\,\Omega(V^{-1}(\phi))\,\mathrm{e}^{-\overline{\phi}\phi} \tag{5}$$

where J is a notation for the Jacobian determinant. The last integral can be rewritten as

$$\int_{\mathbb{C}^m} \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\frac{\Omega(V^{-1}(\phi))}{JV(V^{-1}(\phi))}\,\mathrm{e}^{-\overline{\phi}\phi} = \frac{\Omega(V^{-1}(0))}{JV(V^{-1}(0))} \tag{6}$$

since the Gaussian measure $\mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{e}^{-\overline{\phi}\phi}$ projects a pure holomorphic integrand on its constant term. When applied to

$$\Omega(\phi) \stackrel{\text{def}}{=} \frac{\phi_1^{k_1}}{k_1!}\cdots\frac{\phi_m^{k_m}}{k_m!} \tag{7}$$

the previous computation gives

$$\int_{\mathbb{C}^m} \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\frac{\phi_1^{k_1}}{k_1!}\cdots\frac{\phi_m^{k_m}}{k_m!}\,\mathrm{e}^{-\overline{\phi}\phi+\overline{\phi}xg(\phi)} = \frac{f_1^{k_1}}{k_1!}\cdots\frac{f_m^{k_m}}{k_m!}\times\frac{1}{\det(\delta_{ij}-x_i g_{ij}(f_1,\ldots,f_m))} \tag{8}$$

from which the statement of theorem 1 easily follows by derivation of the left-hand side with respect to the x variables, and evaluation at $x=0$.

The previous derivation could ably serve as an example illustrating the magic of symbolic manipulations when used to prove mathematical identities [3]. However, there are obvious problems in making the previous derivation rigorous, in a functional setting. The first one is that the integral (4) is not convergent. The second one is that the change of variables altered ϕ but left its complex conjugate $\overline{\phi}$ unchanged. These two problems could presumably be avoided, in the case where the coefficients of the g_i as well as the variables x_i are real, by interpreting ϕ and $\overline{\phi}$ as Fourier dual instead of complex conjugate variables. One would also need to put $\sqrt{-1}$ in front of the 'action' $\overline{\phi}\phi - \overline{\phi}xg(\phi)$ and to use the theory of oscillating integrals to give a rigorous meaning to (4) (see the closely related exercise 3.2 in [10]). However one would then obtain, at best, an *analytic* proof of the Lagrange–Good formula, i.e. what the original proof using complex analysis by Good [8] already was. A recurrent theme in mathematics is trying to find purely *algebraic* proofs of theorems that were first obtained through *transcendental* means. Combinatorialists go even further in their requirements: they want *bijective* proofs, where, for instance, an equality of generating functions is understood in terms of an *explicit* correspondence between combinatorial objects counted in both sides of the equation [9, 17]. Combinatorialists sometimes have to work hard to obtain this improvement [7] over an analytic proof [8]. In our case, and *this is the main point of the present paper*, no work is needed: the translation from the functional setting to the combinatorial one is *automatic* and simply is the standard perturbation expansion into Feynman diagrams that one learns among the very basics of quantum field theory. Let us also point out that such diagrammatic techniques are not so well known, or trusted, among mathematicians, even combinatorialists. We therefore hope that this work will provide the latter with some incentive to learn the modicum of quantum field theory represented by these diagrammatic techniques which account, in a large part, for the language barrier between mathematicians and physicists. An effort in this direction is made in [2] where we provide, using Joyal's theory of combinatorial species [12], a framework for the use of Feynman diagrammatic expansions as a *reliable* tool for proving theorems in mathematics, even when the *safety net* of a functional interpretation is not available (see, for instance, our diagrammatic proof of the Cayley–Hamilton theorem in section III.3.3 of [1]). Last but not least, let us also add that the diagrammatic version of our proof, given in the next section, is also *rigorous*. Indeed, all the diagrammatic sums we will write are easily seen to be convergent in the ring $\mathbb{C}[[x_1, \ldots, x_m]]$ of formal power series with its usual topology.

2. The proof

First, we avoid the use of multi-indices and write

$$g_i(x_1, \ldots, x_m) = \sum_{d\geqslant 0} \frac{1}{d!} \sum_{\alpha_1,\ldots,\alpha_d=1}^{m} w^{[d]}_{i,\alpha_1\ldots\alpha_d} x_{\alpha_1} \ldots x_{\alpha_d} \tag{9}$$

where the tensor element $w^{[d]}_{i,\alpha_1\ldots\alpha_d}$ is completely symmetric in $\alpha_1, \ldots, \alpha_d$. Therefore the $f_i(x_1, \ldots, x_m)$ are the solutions of

$$f_i = x_i \sum_{d\geqslant 0} \frac{1}{d!} \sum_{\alpha_1,\ldots,\alpha_d=1}^{m} w^{[d]}_{i,\alpha_1\ldots\alpha_d} f_{\alpha_1} \ldots f_{\alpha_d} \tag{10}$$

which can be rewritten as the direct reversion problem, with unknowns $f_1, \ldots, f_m$,

$$y_i = \Gamma_i(f) \tag{11}$$

with $y_i \stackrel{\text{def}}{=} x_i w^{[0]}_i$ and

$$\Gamma_i(f) \stackrel{\text{def}}{=} f_i - \sum_{d\geqslant 1} \sum_{\alpha_1,\ldots,\alpha_d=1}^{m} \frac{1}{d!} x_i w^{[d]}_{i,\alpha_1\ldots\alpha_d} f_{\alpha_1} \ldots f_{\alpha_d} \tag{12}$$

that is

$$\Gamma_i(f) = \sum_{d\geqslant 1} \frac{1}{d!} \sum_{\alpha_1,\ldots,\alpha_d=1}^{m} \eta^{[d]}_{i,\alpha_1\ldots\alpha_d} f_{\alpha_1}\ldots f_{\alpha_d} \tag{13}$$

with

$$\eta^{[1]}_{i,\alpha} \stackrel{\text{def}}{=} \delta_{i\alpha} - x_i w^{[1]}_{i,\alpha} \qquad \text{for} \quad i, \alpha \in \{1,\ldots,m\} \tag{14}$$

and

$$\eta^{[d]}_{i,\alpha_1\ldots\alpha_d} \stackrel{\text{def}}{=} -x_i w^{[d]}_{i,\alpha_1\ldots\alpha_d} \tag{15}$$

for $d \geqslant 2$, and $i, \alpha_1, \ldots, \alpha_d \in \{1,\ldots,m\}$.

It was shown in [1] (see [2] for more detail) that the solution of such a reversion problem is given by the perturbation expansion of the following quantum field theory one-point function

$$f_i = \frac{\int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\phi_i\,\mathrm{e}^{-\overline{\phi}\Gamma(\phi)+\overline{\phi}y}}{\int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{e}^{-\overline{\phi}\Gamma(\phi)+\overline{\phi}y}}. \tag{16}$$

Here $\overline{\phi}_1,\ldots,\overline{\phi}_m, \phi_1,\ldots,\phi_m$ are the components of a complex Bosonic field. The integration is over $\mathbb{C}^m$ with the measure

$$\mathrm{d}\overline{\phi}\,\mathrm{d}\phi \stackrel{\text{def}}{=} \prod_{i=1}^{m} \left(\frac{\mathrm{d}(\mathrm{Re}\,\phi_i)\,\mathrm{d}(\mathrm{Im}\,\phi_i)}{\pi} \right) \tag{17}$$

we used the notation $\overline{\phi}\Gamma(\phi) \stackrel{\text{def}}{=} \sum_{i=1}^m \overline{\phi}_i \Gamma_i(\phi_1,\ldots,\phi_m)$, and $\overline{\phi}y \stackrel{\text{def}}{=} \sum_{i=1}^m \overline{\phi}_i y_i$. If $\Omega(\overline{\phi},\phi)$ is a function of the fields, we use the notation

$$\langle \Omega(\overline{\phi},\phi)\rangle_U \stackrel{\text{def}}{=} \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\Omega(\overline{\phi},\phi)\,\mathrm{e}^{-\overline{\phi}\Gamma(\phi)+\overline{\phi}y} \tag{18}$$

for the corresponding *unnormalized* correlation function, and

$$\langle \Omega(\overline{\phi},\phi)\rangle_N \stackrel{\text{def}}{=} \frac{1}{Z} \langle \Omega(\overline{\phi},\phi)\rangle_U \tag{19}$$

for the corresponding *normalized* correlation function, where the normalization factor is

$$Z \stackrel{\text{def}}{=} \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{e}^{-\overline{\phi}\Gamma(\phi)+\overline{\phi}y}. \tag{20}$$

Finally we denote by $\langle\cdots\rangle_C$ the *connected* correlation functions, also known as *cumulants* or *semi-invariants* in mathematical statistics and probability theory.

Note that the 'action' $S(\overline{\phi},\phi) \stackrel{\text{def}}{=} \overline{\phi}\Gamma(\phi) - \overline{\phi}y$ in the exponential can be separated into quadratic and nonquadratic parts by writing

$$\Gamma(\phi) = C^{-1}\phi - H(\phi) \tag{21}$$

with

$$[C^{-1}]_{ij} \stackrel{\text{def}}{=} \eta^{[1]}_{i,j} = \delta_{ij} - x_i w^{[1]}_{i,j} \tag{22}$$

and

$$H(\phi) \stackrel{\text{def}}{=} \sum_{d\geqslant 2} \sum_{\alpha_1,\ldots,\alpha_d=1}^{m} \left(-\frac{1}{d!}\right) \eta^{[d]}_{i,\alpha_1\ldots\alpha_d} \phi_{\alpha_1}\ldots\phi_{\alpha_d}. \tag{23}$$

C is the free propagator of our theory, $\overline{\phi}H(\phi)$ is the interaction potential and $\overline{\phi}y$ contains the sources which can be treated as particular vertices of the interaction. Therefore

$$\mathrm{e}^{-S(\overline{\phi},\phi)} = \mathrm{e}^{-\overline{\phi}C^{-1}\phi+\overline{\phi}H(\phi)+\overline{\phi}y} \tag{24}$$

and we let

$$d\mu_C(\overline{\phi}, \phi) \stackrel{\text{def}}{=} \frac{d\overline{\phi}\, d\phi}{\det\, C}\, e^{-\overline{\phi}C^{-1}\phi} \tag{25}$$

be the normalized complex Gaussian measure with covariance C. As a result

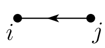
$$Z = \int d\overline{\phi}\, d\phi\, e^{-S(\overline{\phi},\phi)} \tag{26}$$

$$= (\det C) \int d\mu_C(\overline{\phi}, \phi)\, e^{\overline{\phi}H(\phi)+\overline{\phi}y}. \tag{27}$$

Now, by the standard rules of perturbative quantum field theory,

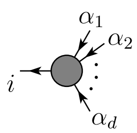
$$\log\left(\int d\mu_C(\overline{\phi}, \phi)\, e^{\overline{\phi}H(\phi)+\overline{\phi}y}\right)$$

is the sum over the connected vacuum Feynman diagrams built using the propagators



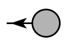
$$= C_{ij} \tag{28}$$

the H-vertices



$$= -\eta^{[d]}_{i,\alpha_1\ldots\alpha_d} \tag{29}$$

with $d \geqslant 2$, and the y-vertices



$$= y_i. \tag{30}$$

These diagrams are made of a single oriented loop of H-vertices linked by free propagators C, on which tree diagrams terminating with y-vertices are hooked. Since the sum over such tree diagrams builds the one-point function $\langle\phi_i\rangle_N = f_i = \Gamma^{-1}(y)$, it is easy to see that

$$\log\left(\int d\mu_C(\overline{\phi}, \phi)\, e^{\overline{\phi}H(\phi)+\overline{\phi}y}\right) = \sum_{k\geqslant 1} \frac{1}{k}\, \mathrm{tr}[C\partial H(\Gamma^{-1}(y))]^k \tag{31}$$

where $\partial H(z)$ is the matrix with entries $\frac{\partial H_i}{\partial z_j}(z)$. Therefore

$$Z = (\det C)\, e^{-\mathrm{tr}\log(I - C\partial H(\Gamma^{-1}(y)))} \tag{32}$$

or

$$Z^{-1} = \det(C^{-1}(I - C\partial H(\Gamma^{-1}(y)))) \tag{33}$$

$$= \det(C^{-1} - \partial H(\Gamma^{-1}(y))). \tag{34}$$

Now note that

$$\partial\Gamma(\phi) = C^{-1} - \partial H(\phi) \tag{35}$$

so

$$Z^{-1} = \det(\partial\Gamma(\Gamma^{-1}(y))) = \det(\partial\Gamma(f)). \tag{36}$$

Now we also have by (12)

$$\Gamma_i(f) = f_i - \left(x_i g_i(f) - x_i w_i^{[0]}\right) \tag{37}$$

and thus

$$[\partial\Gamma(f)]_{ij} = \frac{\partial}{\partial f_j}\left(f_i - x_i g_i(f) + x_i w_i^{[0]}\right) \tag{38}$$

$$= \delta_{ij} - x_i g_{ij}(f) \tag{39}$$

that is

$$Z = \frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} \tag{40}$$

which is our interpretation of the determinantal denominator in the Lagrange–Good formula as a normalization factor for a probability measure.

Besides, (37) can be rewritten as

$$\Gamma_i(f) - y_i = f_i - x_i g_i(f) \tag{41}$$

that is (16) becomes

$$f_i = \frac{1}{Z}\int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\phi_i\,\mathrm{e}^{-\overline{\phi}\phi+\overline{\phi}xg(\phi)} \tag{42}$$

with $\overline{\phi}xg(\phi) \stackrel{\text{def}}{=} \sum_{i=1}^{m}\overline{\phi}_i x_i g_i(\phi)$ and

$$Z = \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{e}^{-\overline{\phi}\phi+\overline{\phi}xg(\phi)}. \tag{43}$$

Now

$$\frac{f_1^{k_1}}{k_1!}\cdots\frac{f_m^{k_m}}{k_m!}\times\frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} = \frac{Z\langle\phi_1\rangle_C^{k_1}\ldots\langle\phi_m\rangle_C^{k_m}}{k_1!\ldots k_m!} \tag{44}$$

but

$$\left\langle\phi_1^{k_1}\ldots\phi_m^{k_m}\right\rangle_N = \langle\phi_1\rangle_C^{k_1}\ldots\langle\phi_m\rangle_C^{k_m} \tag{45}$$

because a connected graph can hook to at most one of the sources ϕ_i. As a result

$$\frac{f_1^{k_1}}{k_1!}\cdots\frac{f_m^{k_m}}{k_m!}\times\frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))} = \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\frac{\phi_1^{k_1}}{k_1!}\cdots\frac{\phi_m^{k_m}}{k_m!}\,\mathrm{e}^{-\overline{\phi}\phi+\overline{\phi}xg(\phi)}. \tag{46}$$

Now it all becomes very simple since, on expanding $\mathrm{e}^{\overline{\phi}xg(\phi)}$, one gets

$$\frac{f_1^{k_1}}{k_1!}\cdots\frac{f_m^{k_m}}{k_m!}\times\frac{1}{\det(\delta_{ij} - x_i g_{ij}(f))}$$
$$= \sum_{n_1,\ldots,n_m=0}^{+\infty}\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_m^{n_m}}{n_m!}\int \mathrm{d}\mu_I(\overline{\phi},\phi)\prod_{a=1}^{m}\left(\frac{\phi_a^{k_a}}{k_a!}\right)\prod_{b=1}^{m}(\overline{\phi}_b g_b(\phi))^{n_b} \tag{47}$$

where $\mathrm{d}\mu_I(\overline{\phi},\phi) \stackrel{\text{def}}{=} \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{e}^{-\overline{\phi}\phi}$, the Gaussian measure with covariance equal to the identity matrix. Now, by integration of the $\overline{\phi}$ by parts,

$$\int \mathrm{d}\mu_I(\overline{\phi},\phi)\prod_{a=1}^{m}\left(\frac{\phi_a^{k_a}}{k_a!}\right)\prod_{b=1}^{m}(\overline{\phi}_b g_b(\phi))^{n_b} = \int \mathrm{d}\mu_I(\overline{\phi},\phi)\Omega(\phi) \tag{48}$$

with

$$\Omega(\phi) \stackrel{\text{def}}{=} \frac{\partial^{n_1+\cdots+n_m}}{\partial\phi_1^{n_1}\ldots\partial\phi_m^{n_m}}\left(\frac{\phi_1^{k_1}}{k_1!}\cdots\frac{\phi_m^{k_m}}{k_m!}g_1(\phi)^{n_1}\ldots g_m(\phi)^{m_1}\right). \tag{49}$$

Since $\Omega(\phi)$ only depends on ϕ, $\int \mathrm{d}\mu_I(\overline{\phi},\phi)\Omega(\phi)$ is equal to the constant term of $\Omega(\phi)$ which is easily seen to be the coefficient of

$$\frac{\phi_1^{n_1}}{n_1!}\cdots\frac{\phi_m^{n_m}}{n_m!} \qquad \text{in} \qquad \frac{\phi_1^{k_1}}{k_1!}\cdots\frac{\phi_m^{k_m}}{k_m!}g_1^{n_1}(\phi)\ldots g_m^{n_m}(\phi)$$

which concludes our proof.

Acknowledgments

The content of this paper is an extension of techniques developed in collaboration with V Rivasseau and presented in [1]. We thank J Feldman for his invitation to the Mathematics Department of the University of British Columbia where part of this work was done. The pictures in this paper have been drawn using a software package that was kindly provided by J Feldman. We also thank the anonymous referee for useful comments that helped to improve the clarity of our presentation. Finally the support of the Centre National de la Recherche Scientifique is gratefully acknowledged.

References

[1] Abdesselam A 2002 The Jacobian conjecture as a problem of perturbative quantum field theory *Preprint* math.CO/0208173

[2] Abdesselam A 2002 Feynman diagrams in algebraic combinatorics *Preprint* math.CO/0212121

[3] Cartier P 2000 Mathemagics (a tribute to L Euler and R Feynman) *Sém. Lothar. Combin.* **44** Art. B44d, 71 pp (electronic)

[4] Darboux G 1869 Sur la série de Laplace *C. R. Acad. Sci., Paris* **68** 324–7

[5] Ehrenborg R and Méndez M 1994 A bijective proof of infinite variated Good's inversion *Adv. Math.* **103** 221–59

[6] Gallavotti G 1994 Perturbation theory *Mathematical Physics Towards the XXI Century* ed R Sen and A Gersten (Ber Sheva: Ben Gurion University Press) pp 275–94

[7] Gessel I M 1987 A combinatorial proof of the multivariable Lagrange inversion formula *J. Combin. Theory* A **45** 178–95

[8] Good I J 1960 Generalization to several variables of Lagrange's expansion, with applications to stochastic processes *Proc. Camb. Phil. Soc.* **56** 367–80

[9] Goulden I P and Jackson D M 1983 *Combinatorial Enumeration* (New York: Wiley)

[10] Grigis A and Sjöstrand J 1994 *Microlocal Analysis for Differential Operators, An introduction, (London Mathematical Society Lecture Notes Series* vol 196*)* (Cambridge: Cambridge University Press)

[11] Jacobi C G J 1830 De resolutione aequationum per series infinitas *J. Reine Angew. Math.* **6** 257–86

[12] Joyal A 1981 Une théorie combinatoire des séries formelles *Adv. Math* **42** 1–82

[13] de Lagrange J L 1770 Nouvelle méthode pour résoudre des équations littérales par le moyen des séries *Mém. Acad. Roy. Sci. Belles-Lettres Berlin* **24**

[14] Laplace P S 1777 Mémoire sur l'usage du calcul aux différences partielles dans la théorie des suites *Mém. Acad. Roy. Sci. Paris* 99–122

[15] MacMahon P A 1915 *Combinatory Analysis* vol 1 (Cambridge: Cambridge University Press)

[16] Poincaré H 1886 Sur les résidus des intégrales doubles *Acta. Math.* **9** 321–80

[17] Stanley R P 1999 *Enumerative Combinatorics* vol 2 (Cambridge: Cambridge University Press)

[18] Stieltjes T J 1885 Sur une généralisation de la série de Lagrange *Ann. Sci. École Normale. Paris Sér.* 3 **2** 93–8